

Shortcut to adiabatic gate teleportation

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We introduce a shortcut to the adiabatic gate teleportation model of quantum computation. More specifically, we determine fast local counter-diabatic Hamiltonians able to implement teleportation as a universal computational primitive. In this scenario, we provide the counter-diabatic driving for arbitrary n -qubit gates, which allows to achieve universality through a variety of gate sets. Remarkably, our approach maps the superadiabatic Hamiltonian H_{SA} for an arbitrary n -qubit *gate* teleportation into the implementation of a rotated superadiabatic dynamics of an n -qubit *state* teleportation. This result is rather general, with the speed of the evolution only dictated by the quantum speed limit. In particular, we analyze the energetic cost for different Hamiltonian interpolations in the context of the energy-time complementarity.

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I. INTRODUCTION

Quantum teleportation [1] is a valuable tool for a number of quantum tasks. In quantum communication, it makes available a quantum channel for transmission of unknown states between two agents (Alice and Bob) separated by a large distance (currently more than 100 km in optical fibers [2] or 143 km in a free-space link [3]). In quantum information processing, quantum teleportation can be applied as a primitive for universal quantum computation (QC), as remarkably shown by Gottesman and Chuang in Ref. [4]. In this approach, a third party (Charlie) provides rotated Bell states to Alice and Bob, who can implement universal QC by solely performing single-qubit operations and Bell measurements. In particular, this method is a precursor of the paradigm of measurement-based QC (see, e.g., Ref. [5]). More recently, QC via quantum teleportation has been formulated via adiabatic evolution by Bacon and Flammia [6], providing a hybrid approach for QC (see also Ref. [7] for an alternative adiabatic hybrid approach). In this scenario, a quantum circuit can be mapped in a sequence of piecewise Hamiltonian evolutions implementing single- and double-gate teleportation protocols, allowing for universality through the set of one-qubit rotations joint with an entangling two-qubit gate [8, 9]. However, since these processes are ruled by the adiabatic approximation, it turns out that each gate of the adiabatic circuit will be implemented within some fixed probability (for a finite evolution time). Moreover, the time for performing each individual gate will be bounded from below by the adiabatic time condition [10].

In order to speed up the adiabatic evolution in the Bacon-Flammia hybrid model, we propose here a general shortcut to adiabatic gate teleportation via counter-diabatic assistant Hamiltonians within the framework of the superadiabatic theory [11–14]. In particular, we introduce the concept of superadiabatic gate teleportation, showing that it can be used as

a fast primitive for universal QC. The use of superadiabatic evolutions for universal QC via local interactions has recently been proposed in Ref. [15], where it is shown how to implement arbitrary n -controlled gates with minima ancilla requirements. The physical resources spent by this strategy will be governed by the quantum circuit complexity, but no adiabatic constraint will be required in the individual implementation of the quantum gates. Moreover, the gates will be deterministically implemented with probability one as long as decoherence effects can be avoided. This analog approach allows for fast implementation of individual gates, whose time consumption is only dictated by the quantum speed limit (QSL) (for closed systems, see Refs. [16–19]). Indeed, the time demanded for each gate will imply an energy cost, which increases with the desired speed of the evolution.

The paper is organized as follows. In Sec. II, we discuss the adiabatic gate teleportation protocol as originally proposed in Ref. [6], by explicitly extending it to arbitrary n -qubit gates. In Sec. III, we derive a shortcut for the adiabatic teleportation of n -qubit gates, showing that it can be used to implement universal QC. Moreover, since no adiabaticity is required, we also analyze the energetic cost for implementing superadiabatic universal QC via adiabatic gate teleportation. Section IV is devoted to our conclusions.

II. UNIVERSAL QC VIA ADIABATIC TELEPORTATION

A. Adiabatic teleportation of one-qubit states

Given an unknown state $|\psi\rangle = a|0\rangle + b|1\rangle$, where $|a|^2 + |b|^2 = 1$, adiabatic teleportation can be implemented through the Hamiltonian [6]

$$H_0(s) = \eta_i(s) H_i + \eta_f(s) H_f, \quad (1)$$

where $\eta_i(0) = \eta_f(1) = 1$, $\eta_i(1) = \eta_f(0) = 0$, and

$$H_i = -\omega\hbar (\mathbb{1}XX + \mathbb{1}ZZ), \quad (2)$$

$$H_f = -\omega\hbar (XX\mathbb{1} + ZZ\mathbb{1}), \quad (3)$$

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where X and Z are Pauli spin- $\frac{1}{2}$ operators and $s = t/\tau$ is the normalized time, with τ the total evolution time. The state of the system at $t = 0$ is prepared as $|\phi(0)\rangle = (1/\sqrt{2})|\psi\rangle(|00\rangle + |11\rangle)$. To prove that teleportation happens, we must show that the final state of the system is given by $(1/\sqrt{2})(|00\rangle + |11\rangle)|\psi\rangle$. A scheme of the process is shown in Fig. 1.

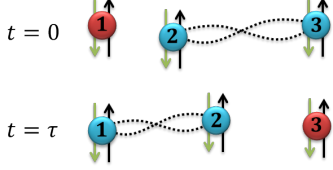


FIG. 1. (Color online) Adiabatic teleportation of a single qubit (red particle). The quantum state initially encoded in qubit 1 ($t = 0$) is teleported to qubit 3 ($t = \tau$), with a Bell pair (blue particles) used as a resource for the protocol.

It is important to notice that the Hamiltonian $H_0(s)$ acts only on qubits 2 and 3 for $s = 0$, only on qubits 1 and 2 for $s = 1$, and on all the 3 qubits for $0 < s < 1$. Since $H_0(s)$ is doubly degenerate, the adiabatic theorem implies solely in the decoupled evolution of the eigenspaces of $H_0(s)$. Then, in order to show the success of the adiabatic teleportation via $H_0(s)$, Bacon and Flammia [6] proceeded by developing an analysis based on logical qubits. Here, we devise an alternative derivation, which is based directly on the symmetries of $H_0(s)$. First, consider the commutation relations

$$[H_0(s), \Pi_z] = [H_0(s), \Pi_x] = 0, \quad (4)$$

with $\Pi_z = ZZZ$ and $\Pi_x = XXX$. For a state of the computational basis $\{|nmk\rangle\}$ we have

$$\Pi_z |nmk\rangle = (-1)^{n+m+k} |nmk\rangle, \quad (5)$$

$$\Pi_x |nmk\rangle = |\bar{n}\bar{m}\bar{k}\rangle, \quad (6)$$

where we have defined $|\bar{0}\rangle \equiv |1\rangle$ and $|\bar{1}\rangle \equiv |0\rangle$. Notice that Π_z and Π_x are parity operators, each of them associated with a Z_2 symmetry of the Hamiltonian. Now, let us define the sets $\{|nmk\rangle_{\pm}\}$ given by vectors of the computational basis with Π_z eigenvalues ± 1 . Then, from the commutation of the Hamiltonian $H_0(s)$ with Π_z , we obtain that parity is conserved throughout the evolution, which means that we can conveniently write $H_0(s)$ in a block-diagonal basis

$$H_0(s) = \begin{pmatrix} H_{4 \times 4}^+(s) & \mathbb{0}_{4 \times 4} \\ \mathbb{0}_{4 \times 4} & H_{4 \times 4}^-(s) \end{pmatrix}, \quad (7)$$

where the basis has been ordered in terms of $\{|nmk\rangle_+, |nmk\rangle_-\}$. In addition, the symmetry Π_x ensures a relationship between the elements of $H_{4 \times 4}^+(s)$ and $H_{4 \times 4}^-(s)$ so that, if we conveniently sort the computational basis in the parity subspaces $\{|nmk\rangle_+\}$ and $\{|nmk\rangle_-\}$, we find that $H_{4 \times 4}^+(s) = H_{4 \times 4}^-(s)$. In fact, by computing the matrix elements of $H_{4 \times 4}^+(s)$ and $H_{4 \times 4}^-(s)$ and by using that $\Pi_x |nmk\rangle_+ = |\bar{n}\bar{m}\bar{k}\rangle_-$, we get

$$-\langle \bar{n}' \bar{m}' \bar{k}' | H_0(s) | \bar{n} \bar{m} \bar{k} \rangle_- = + \langle n' m' k' | H_0(s) | nmk \rangle_+. \quad (8)$$

Then, by computing the spectrum of $H_{4 \times 4}^{\pm}(s)$, we completely determine the spectrum of $H_0(s)$. More specifically, the energies associated with $H_{4 \times 4}^{\pm}(s)$ read as

$$E_0(s) = -2\omega\hbar \sqrt{\eta_i^2(s) + \eta_f^2(s)}, \quad (9)$$

$$E_1(s) = E_2(s) = 0, \quad (10)$$

$$E_3(s) = 2\omega\hbar \sqrt{\eta_i^2(s) + \eta_f^2(s)}, \quad (11)$$

with the gap between the ground state and the first excited state given by

$$\varepsilon(s) = 2\omega\hbar \sqrt{\eta_i^2(s) + \eta_f^2(s)}. \quad (12)$$

We can observe that $\varepsilon(s) \neq 0 \forall s \in [0, 1]$ because $\eta_i(s)$ and $\eta_f(s)$ never simultaneously vanish. To conclude the teleportation of the initial state, it remains to show that the final state of the third qubit is exactly $|\psi\rangle$. To this end, let us write the initial and final states as

$$|\phi(0)\rangle = \frac{1}{\sqrt{2}} (a|0\rangle_1 + b|1\rangle_1) (|00\rangle_{23} + |11\rangle_{23}), \quad (13)$$

$$|\phi(1)\rangle = \frac{1}{\sqrt{2}} (|00\rangle_{12} + |11\rangle_{12}) (\alpha|0\rangle_3 + \beta|1\rangle_3), \quad (14)$$

where the form of $|\phi(1)\rangle$ is ensured by the adiabatic theorem, with general coefficients $\alpha = \alpha(a, b)$ and $\beta = \beta(a, b)$. Now notice that Eq. (13) implies that the coefficients a and b multiply the states of parity +1 and -1, respectively. In addition, Eq. (14) implies that the coefficients $\alpha(a, b)$ and $\beta(a, b)$ also multiply states of parity +1 and -1, respectively. Due to the symmetry Π_z , it follows that states of different parities evolve independently. Then, $\alpha = \alpha(a)$ and $\beta = \beta(b)$. Moreover, since the evolution of the system is unitary, we have that $\langle \phi(0) | \phi(0) \rangle = \langle \phi(1) | \phi(1) \rangle = 1$. This implies that $|\alpha(a)|^2 = |a|^2$ and $|\beta(b)|^2 = |b|^2$. Consequently, $\alpha(a) = ae^{i\theta_a}$ and $\beta(b) = be^{i\theta_b}$, for any θ_a and θ_b real. On the other hand, we can use the parity Π_x to show that states of parities +1 and -1 have identical evolution, since $H_{4 \times 4}^+(s) = H_{4 \times 4}^-(s)$. Then, $\theta_a = \theta_b = \theta$. Hence,

$$|\phi(1)\rangle = \frac{1}{\sqrt{2}} (|00\rangle_{12} + |11\rangle_{12}) (a|0\rangle_3 + b|1\rangle_3), \quad (15)$$

up to a global phase $e^{i\theta}$. This concludes the proof of the adiabatic teleportation of a single qubit.

B. Adiabatic teleportation of n -qubit states

Let us begin by generalizing the previous protocol to implement now the adiabatic teleportation of an unknown two-qubit state. In this direction, we will consider a quantum system composed of six qubits. A scheme of the process is exhibited in Fig. 2. The composite state to be teleported is prepared in qubits 1 and 2 and the final state in qubits 5 and 6, with two Bell pairs used as the resource for the protocol. Let us write the state to be teleported as

$$|\psi\rangle_{12} = \alpha|00\rangle_{12} + \delta|01\rangle_{12} + \gamma|10\rangle_{12} + \beta|11\rangle_{12}, \quad (16)$$

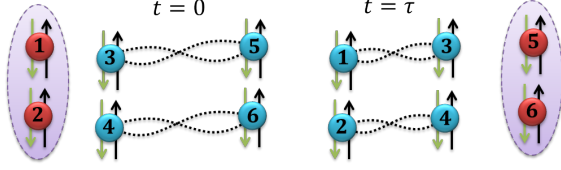


FIG. 2. (Color online) Adiabatic teleportation of a two-qubit state (red particles). The composite state to be teleported is prepared in qubits 1 and 2, with the final state of teleportation in qubits 5 and 6. Two Bell pairs (blue particles) are used in the protocol.

The adiabatic teleportation of the initial state will be performed through the Hamiltonian

$$H_D(s) = \mathbb{1}_{\text{even}} \otimes H_{\text{odd}}(s) + H_{\text{even}}(s) \otimes \mathbb{1}_{\text{odd}}, \quad (17)$$

where $H_{\text{even}}(s)$ and $H_{\text{odd}}(s)$ are given by $H_0(s)$ as given by Eq. (1) acting over qubits labeled with *even* and *odd* indices, respectively. Then, no interaction between the odd and even sectors will occur. To determine the spectrum of $H_D(s)$ we will make use of the following general result: Let us consider $A_{m \times m}$ and $B_{n \times n}$ as two operators such that $A_{m \times m} |a_\mu\rangle = a_\mu |a_\mu\rangle$ and $B_{n \times n} |b_\eta\rangle = b_\eta |b_\eta\rangle$, with the sets of eigenvalues $\{b_\eta\}$ and $\{a_\mu\}$ associated with the eigenvector bases $\{|b_\eta\rangle\}$ and $\{|a_\mu\rangle\}$, respectively. Thus, if we consider an operator $C_{k \times k}$, where $k = mn$, such that $C_{k \times k} = A_{m \times m} \otimes \mathbb{1}_{n \times n} + \mathbb{1}_{m \times m} \otimes B_{n \times n}$, then $|c_{\mu\eta}\rangle = |a_\mu\rangle \otimes |b_\eta\rangle$ are the eigenvectors of $C_{k \times k}$ associated with the eigenvalues $c_{\mu\eta} = a_\mu + b_\eta$. Bearing in mind this result, the spectrum of $H_D(s)$ is simply given by

$$E_{kl}(s) = E_k^{\text{odd}}(s) + E_l^{\text{even}}(s), \quad (18)$$

where $E_k^{\text{odd}}(s)$ and $E_k^{\text{even}}(s)$ are given by Eqs. (9), (10), and (11). By using Eq. (18), we show that the gap of the $H_D(s)$ is $\varepsilon_D(s) = E_{01}(s) - E_{00}(s) = \varepsilon(s)$, where $\varepsilon(s)$ was determined by Eq. (12). As each sector has the symmetries Π_x and Π_z , we define the operators

$$\begin{aligned} \Pi_{z \text{ odd}} &\equiv \mathbb{1} \otimes \Pi_z, \quad \Pi_{z \text{ even}} \equiv \Pi_z \otimes \mathbb{1}, \\ \Pi_{x \text{ odd}} &\equiv \mathbb{1} \otimes \Pi_x, \quad \Pi_{x \text{ even}} \equiv \Pi_x \otimes \mathbb{1}, \end{aligned}$$

where the left operators in the tensor product act on the even sector, with the right operators acting on the odd sector. It then follows that these operators (and their products) are Z_2 symmetries of $H_D(s)$. Considering the symmetry operator $\Pi_z^D = \Pi_{z \text{ even}} \Pi_{z \text{ odd}}$, we then write

$$H_D(s) = \begin{pmatrix} H_{32 \times 32}^+(s) & 0 \\ 0 & H_{32 \times 32}^-(s) \end{pmatrix}, \quad (19)$$

where $H_{32 \times 32}^\pm(s)$ acts on the states of parity ± 1 of the operator Π_z^D . By using now the symmetry $\Pi_x^D = \Pi_{x \text{ even}} \Pi_{x \text{ odd}}$, we can choose the order of the basis such that $H_{32 \times 32}^+(s) = H_{32 \times 32}^-(s)$. In addition, by using the symmetries $\Pi_{z \text{ odd}}$ and $\Pi_{z \text{ even}}$ of each sector we get

$$H_D(s) = \begin{pmatrix} H_\alpha(s) & 0 & 0 & 0 \\ 0 & H_\beta(s) & 0 & 0 \\ 0 & 0 & H_\gamma(s) & 0 \\ 0 & 0 & 0 & H_\delta(s) \end{pmatrix}, \quad (20)$$

where we have considered the specific parity ordering $\{|E\rangle_+ |O\rangle_+, |E\rangle_- |O\rangle_-, |E\rangle_- |O\rangle_+, |E\rangle_+ |O\rangle_-\}$ in the computational basis, with the definitions $|E\rangle \equiv |n_2 n_4 n_6\rangle$ and $|O\rangle \equiv |n_1 n_3 n_5\rangle$. Moreover, by using the symmetries of $H_D(s)$ with respect to $\Pi_{x \text{ odd}}$ and $\Pi_{x \text{ even}}$, we find that the blocks $\{H_\alpha(s), H_\beta(s), H_\gamma(s), H_\delta(s)\}$ are identical by a suitable organization of the basis vectors.

To show that double teleportation can indeed be adiabatically implemented via the Hamiltonian $H_D(s)$, let us denote the initial and final states as given by

$$|\phi(0)\rangle = |\psi\rangle_{12} |\beta_{00}\rangle_{35} |\beta_{00}\rangle_{46}, \quad (21)$$

$$|\phi(1)\rangle = |\beta_{00}\rangle_{12} |\beta_{00}\rangle_{13} |\tilde{\psi}\rangle_{56}, \quad (22)$$

where $|\beta_{00}\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ and $|\tilde{\psi}\rangle_{56}$ reads as

$$|\psi\rangle_{56} = \tilde{\alpha}|00\rangle_{56} + \tilde{\delta}|01\rangle_{56} + \tilde{\gamma}|10\rangle_{56} + \tilde{\beta}|11\rangle_{56}. \quad (23)$$

Note that, since $H_D(s)$ is degenerate, we cannot associate $|\psi\rangle_{56}$ directly to $|\psi\rangle_{12}$. However, Eq. (20) implies into a dynamics such as $\tilde{\xi} = \tilde{\xi}(\xi)$, where $\tilde{\xi} = \{\tilde{\alpha}, \tilde{\delta}, \tilde{\gamma}, \tilde{\beta}\}$ and $\xi = \{\alpha, \delta, \gamma, \beta\}$. This is because each element of the set $\{\alpha, \delta, \gamma, \beta\}$ is in a distinct parity sector. Moreover, unitarity of the evolution leads to $|\tilde{\xi}(\xi)|^2 = |\xi|^2$, which yields $\tilde{\xi} = \xi e^{i\varphi_\xi}$. By using now the parity operators $\Pi_{x \text{ odd}}$ and $\Pi_{x \text{ even}}$, we can show that the blocks in the Hamiltonian provided by Eq. (20) are identical (by suitably ordering the basis) so that the parameters φ_ξ are globally defined, namely, $\varphi_\xi \equiv \varphi (\forall \xi)$. Hence, we conclude that the state of the qubits 5 and 6 at the final of the process reads as

$$|\psi\rangle_{56} = \alpha|00\rangle_{56} + \delta|01\rangle_{56} + \gamma|10\rangle_{56} + \beta|11\rangle_{56}, \quad (24)$$

up to the global phase $e^{i\varphi}$. We can extend this protocol to perform teleportation of an unknown state of n qubits. In this direction, we need to increase the number of sectors and define a Hamiltonian given by $H_{\text{mult}}(t) = \sum_{k=1}^n H_k(t)$, where each $H_k(t)$ is given by Eq. (1), which acts on an individual sector composed by three qubits. Consequently, n Bell pairs will be used as a resource for the process. A scheme of such generalized protocol is presented in Fig. 3. The Hamiltonian $H_{\text{mult}}(t)$ displays a $2n$ -fold degenerate ground state, which decouples from the rest of the spectrum in the adiabatic dynamics. Teleportation of the n -qubit state will then follow from the z and x parity symmetries in each individual sector.

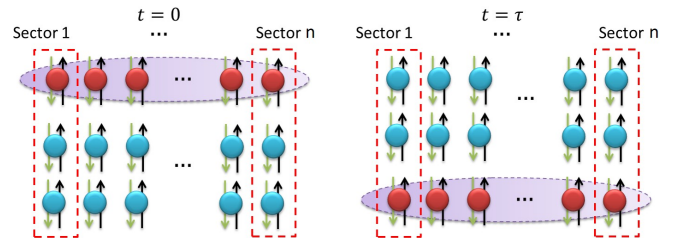


FIG. 3. (Color online) Adiabatic teleportation of an n -qubit state. In each three-qubit sector we have a qubit to be teleported (red particle) and a Bell pair (blue particles).

C. Adiabatic teleportation of unitary n -qubit gates

In the one-qubit gate teleportation protocol, Alice starts with an unknown state $|\psi\rangle$ at qubit 1 and shares a rotated Bell pair $U_3|\beta_{00}\rangle_{23}$ with Bob (prepared by a third party Charlie). Then, by applying the usual teleportation procedure, Bob receives $U_3|\psi\rangle$ at the end of the protocol with probability one as long as decoherence can be neglected. In order to implement the adiabatic version of gate teleportation, we define the gate to be implemented over qubit 3 as $U = \mathbb{1}_1\mathbb{1}_2U_3$, where $U_3^\dagger U_3 = \mathbb{1}_3$. Then, as shown in Ref. [6], the time-dependent Hamiltonian $H_0(s, U)$ able to adiabatically implement the teleportation of the gate U can be determined from the original Hamiltonian $H_0(s)$ for one-qubit teleportation through the rotation

$$H_0(s, U) = UH_0(s)U^\dagger. \quad (25)$$

Indeed, this can be understood directly from the symmetries of $H_0(s, U)$. Since commutation relations are preserved by rotations [20], $H_0(s, U)$ is Z_2 -symmetric under the parity operators $\Pi_z(U) = ZZ(U_3ZU_3^\dagger)$ and $\Pi_x(U) = XX(U_3XU_3^\dagger)$. Then, we can show the teleportation of the gate U by working the computational basis rotated by U . In this new basis, the matrix form of $H_0(s, U)$ is identical to that of the original $H_0(s)$, which implies that the same argument used to the simple teleportation performed by $H_0(s)$ is applicable to case of the Hamiltonian $H_0(s, U)$. The gap of $H_0(s, U)$ is also given by (12) because the spectrum of the operator will not change by a unitary transformation [20]. Hence, the initial state $|\phi(0, U)\rangle = |\psi\rangle_1 U_3|\beta_{00}\rangle_{23}$ (with the rotated Bell pair provided by Charlie) will be adiabatically evolved into the final state $|\phi(1, U)\rangle = |\beta_{00}\rangle_{12} U_3|\psi\rangle_3$.

In order to perform universal QC via adiabatic teleportation, Ref. [6] specifically worked out a Hamiltonian to adiabatically implement the teleportation of the controlled-phase gate. Here, we extend the protocol to adiabatically implement an arbitrary n -qubit unitary gate. By focusing first on two-qubit gates, we use as a fundamental resource the double teleportation protocol, as described in Sec. II B. More specifically, we can show that any two-qubit gate U can be implemented by the Hamiltonian

$$H_D(s, U) = UH_D(s)U^\dagger,$$

where $H_D(s)$ is provided by Eq. (17) and $U = U_{56}$ is the gate to be performed at the final time in the qubits of the Bob. As in the case of single qubits, we have that the spectra of $H_D(s, U)$ and $H_D(s)$ are identical. Then, to show that the two-qubit gate teleportation takes place through the adiabatic dynamics dictated by $H_D(s, U)$, we make use of the following rotated parity symmetry operators:

$$\begin{aligned} \Pi_{z \text{ sec}}(U) &= U\Pi_{z \text{ sec}}^D U^\dagger, & \Pi_{x \text{ sec}}(U) &= U\Pi_{x \text{ sec}}^D U^\dagger, \\ \Pi_x(U) &= U\Pi_x^D U^\dagger, & \Pi_z(U) &= U\Pi_z^D U^\dagger, \end{aligned}$$

where $\text{sec} = \{\text{even}, \text{odd}\}$. Bearing in mind that Charlie provides rotated Bell pairs, we have at $s = 0$ the initial state $|\phi(0, U)\rangle = U_{56}|\phi(0)\rangle$, where $|\phi(0)\rangle$ is given by Eq. (21). In

the rotated basis, the matrix form of $H_D(s, U)$ is also equivalent to the matrix form of $H_D(s)$ as given in the original basis, from which it follows that at the final of the process the state of the system will be $|\phi(1, U)\rangle = U_{56}|\phi(1)\rangle$, where $|\phi(1)\rangle$ is given by Eq. (24). Concerning the adiabatic teleportation of an n -qubit gate U_n , it can be implemented from the simple adiabatic teleportation of an n -qubit state, as previously described. The Hamiltonian that adiabatically implements this task is then $H_{\text{mult}}(t, U_n) = U_n H_{\text{mult}}(t) U_n^\dagger$. This allows for universal QC by using a variety of sets of universal gates, e.g., the set composed by Hadamard added by three-qubit Toffoli gates [21, 22].

III. SUPERADIABATIC QC VIA TELEPORTATION

A. Shortcut to adiabaticity

We can obtain fast piecewise implementation of quantum gates via shortcuts to adiabaticity [11–14], whose evolution time will not be constrained by the adiabatic theorem. We begin by defining the evolution operator

$$U(t) = \sum_n e^{-\frac{i}{\hbar} \int_0^t d\tau E_n(\tau)} e^{-\int_0^t d\tau \langle n|\partial_\tau|n\rangle} |n(t)\rangle \langle n(0)|, \quad (26)$$

where $\{|n(t)\rangle\}$ denotes the instantaneous eigenstate basis of a general time-dependent Hamiltonian $H_0(t)$. The evolution operator $U(t)$ leads an initial state $|\psi(0)\rangle = |n(0)\rangle$ into an evolved state $|\psi(t)\rangle$ given by

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} \int_0^t d\tau E_n(\tau)} e^{-\int_0^t d\tau \langle n|\partial_\tau|n\rangle} |n(t)\rangle, \quad (27)$$

which mimics the adiabatic evolution of $H_0(t)$. Remarkably, such an evolution can be dictated with no adiabatic constraint by the *superadiabatic* Hamiltonian $H_{\text{SA}}(t)$, which reads as

$$H_{\text{SA}}(t) = H_0(t) + H_{\text{CD}}(t), \quad (28)$$

where the additional term $H_{\text{CD}}(t)$ is known as the *counter-diabatic* Hamiltonian. This contribution is shown to be [11–14]

$$H_{\text{CD}}(t) = i\hbar \sum_n (|\partial_t n\rangle \langle n| + \langle \partial_t n|n\rangle |n\rangle \langle n|), \quad (29)$$

where $|\partial_t n\rangle$ is the time derivative of $|n(t)\rangle$. In particular, we have $\langle \partial_t n|n\rangle = 0$ in Eq. (29) for real Hamiltonians. We observe that the terminology *superadiabaticity* has originally been introduced by Berry in Ref. [23] (see also Ref. [24]) as a systematic procedure of adiabatic iterations, aiming at producing successive adiabatic approximants in processes with finite slowness. Here, we use the term *superadiabatic* Hamiltonian in a different scenario, which means a Hamiltonian capable to yield a shortcut to adiabaticity through the presence of a counter-diabatic driving (see Ref. [25, 26] for a comparison between these two approaches).

Note that a superadiabatic implementation of an arbitrary evolution involves the knowledge of the eigenstates of the adiabatic Hamiltonian $H_0(t)$. In some situations, this can be implemented in realizable settings. For instance, there have been

driving protocols proposed for assisted evolutions in quantum critical phenomena [27–29]. On the other hand, as a shortcut to accelerate QC, the application of superadiabaticity is challenging. Here, as we shall see, the superadiabatic implementation of gate teleportation as a primitive for universal QC can be promptly achieved, since we deal with the eigenspectrum of piecewise Hamiltonians, which act over a few qubits.

B. Superadiabatic teleportation of n -qubit states

To derive the superadiabatic version of the teleportation of n -qubit states, we need to determine the counter-diabatic Hamiltonian $H_{\text{CD}}(s)$ associated with the Hamiltonian $H_0(s)$ as given by Eq. (1). By evaluating the eigenstates of the blocks $H_{4 \times 4}^\pm(s)$ in Eq. (7), we get

$$|E_0^\pm(s)\rangle = \left(\frac{\eta_i + \chi}{\eta_f}, \frac{[\chi - \eta_f][\chi + \eta_i]}{\eta_i \eta_f}, \frac{\chi - \eta_f}{\eta_i}, 1 \right), \quad (30)$$

$$|E_1^\pm(s)\rangle = \left(\frac{\eta_i}{\eta_f} - 1, -\frac{\eta_i}{\eta_f}, 0, 1 \right), \quad (31)$$

$$|E_2^\pm(s)\rangle = \left(-\frac{\eta_i}{\eta_f}, \frac{\eta_i}{\eta_f} + 1, 1, 0 \right), \quad (32)$$

$$|E_3^\pm(s)\rangle = \left(\frac{\eta_i - \chi}{\eta_f}, \frac{\eta_f - \eta_i + \chi}{\eta_i - \eta_f + \chi}, -\frac{\eta_f + \chi}{\eta_i}, 1 \right), \quad (33)$$

where $\eta = \eta(s)$ and $|E_n^\pm(s)\rangle$ are the non-normalized eigenstates of $H_{4 \times 4}^\pm(s)$, with the function χ defined as $\chi = \chi(s) \equiv \sqrt{\eta_i^2(s) + \eta_f^2(s)}$. The counter-diabatic Hamiltonian $H_{\text{CD}}(s)$ can now be found by observing that the Z_2 symmetries of the adiabatic Hamiltonian remain in the superadiabatic theory. We enunciate this result by establishing the theorem following (the proof is in the Appendix A).

Theorem 1 Consider a time-dependent Hamiltonian $H_0(t)$ such that $[H_0(t), \Pi_z] = 0$ and $[H_0(t), \Pi_x] = 0$, where Π_z and Π_x are z and x parity operators, respectively. Then, the superadiabatic Hamiltonian $H_{\text{SA}}(t)$ associated with $H_0(t)$ also satisfies $[H_{\text{SA}}(t), \Pi_z] = 0$ and $[H_{\text{SA}}(t), \Pi_x] = 0$.

From Theorem 1 we can write

$$H_{\text{SA}}(s) = \begin{bmatrix} H_{\text{SA}}^+(s) & \emptyset \\ \emptyset & H_{\text{SA}}^-(s) \end{bmatrix}, \quad (34)$$

with $H_{\text{SA}}^\pm(s) \equiv H_{4 \times 4}^\pm(s) + H_{\text{CD}}^\pm(s)$ and $H_{\text{SA}}^+(s) = H_{\text{SA}}^-(s)$. Since the set $\{|E_n^\pm(s)\rangle\}$ is real, we can write the counter-diabatic Hamiltonian as

$$H_{\text{CD}}^\pm(s) = i \frac{\hbar}{\tau} \sum_{n=0}^3 |\partial_s E_n^\pm(s)\rangle \langle E_n^\pm(s)| \quad (35)$$

Now, let us move on to the implementation of the superadiabatic double teleportation. To this end, we consider a general time-dependent Hamiltonian $H_0(s)$, which is split out as

$$H_0(s) = H_0^A(s) \otimes \mathbb{1}^B + \mathbb{1}^A \otimes H_0^B(s), \quad (36)$$

where $H_0^A(s)$ and $H_0^B(s)$ are associated with piecewise superadiabatic Hamiltonians given by $H_{\text{SA}}^A(s)$ and $H_{\text{SA}}^B(s)$, respectively. Thus, we can write

$$H_{\text{SA}}(s) = H_{\text{SA}}^A(s) \otimes \mathbb{1}^B + \mathbb{1}^A \otimes H_{\text{SA}}^B(s). \quad (37)$$

As a consequence, by taking the Hamiltonian of the double teleportation as given by Eq. (17), we have that the superadiabatic Hamiltonian for the double teleportation is

$$H_{\text{SA}}^D(s) = \mathbb{1}_{\text{even}} \otimes H_{\text{SA}}^{\text{odd}}(s) + H_{\text{SA}}^{\text{even}}(s) \otimes \mathbb{1}_{\text{odd}},$$

where $H_{\text{SA}}^{\text{odd}}(s)$ and $H_{\text{SA}}^{\text{even}}(s)$ are the superadiabatic Hamiltonians for each parity sector. Extension for the teleportation of n -qubit states can be achieved by adding more Z_2 -symmetry sector, with the superadiabatic Hamiltonian given by $H_{\text{mult}}^{\text{SA}} = \sum_{k=1}^n (\otimes_{i=1}^{k-1} \mathbb{1}_i) \otimes H_k^{\text{SA}}(t) \otimes (\otimes_{j=k+1}^N \mathbb{1}_j)$, where $H_k^{\text{SA}}(t)$ denotes the superadiabatic Hamiltonian associated with $H_k(t)$, with each $H_k(t)$ [given by Eq. (1)] acting on an individual sector composed by three qubits.

C. Superadiabatic teleportation of n -qubit gates

In order to perform superadiabatic universal QC we need to show how to implement unitaries of one and two qubits with this model. To this end, we devise the the following theorem (the proof is given in the Appendix B).

Theorem 2 Consider two time-dependent Hamiltonians $H_0(t)$ and $H_0(t, G)$ such that $H_0(t, G) = G H_0(t) G^\dagger$, with G denoting a unitary transformation. Then, the superadiabatic Hamiltonian associated with $H_0(t, G)$ can be written as

$$H_{\text{SA}}(t, G) = G H_{\text{SA}}(t) G^\dagger \quad (38)$$

where $H_{\text{SA}}(t)$ is the superadiabatic Hamiltonian of $H_0(t)$.

Since Theorem 2 holds for any unitary operator G and any time-dependent Hamiltonian, we can use it to superadiabatically implement any unitary transformation of n qubits. In particular, by focusing on one and two qubit gates, we can realize universal QC whose primitives are fast local Hamiltonians. For instance, to implement a one-qubit gate teleportation, the superadiabatic Hamiltonian $H_{\text{SA}}(t)$ is given by Eq. (34), while for the case of gate teleportation of two qubits we must consider $H_{\text{SA}}(t)$ such as given by Eq. (37). An important point is that, in the case of superadiabatic evolutions for rotated systems, the initial state is also required to be rotated (by the third party Charlie) so that the final state contains the teleported gate.

D. Energetic cost of superadiabatic gate teleportation

The shortcut via a counter-diabatic Hamiltonian can yield an evolution that is faster than the adiabatic dynamics, but how much faster? This question has been answered for a general superadiabatic evolution in Ref. [15] through the analysis of

the quantum speed limit (QSL) bounds [16–19] applied to superadiabatic dynamics. In particular, as shown in Ref. [15], the total time τ in superadiabatic evolutions can be arbitrarily reduced for any initial and final states as long as energy is injected in the system. More specifically, we may have $\tau\omega \rightarrow 0$, with ω denoting the energy scale of the system. To quantify the expense of energy in a superadiabatic evolution, we adopt the cost measure (see also Refs. [30, 31])

$$\Sigma(\tau) = \frac{1}{\tau} \int_0^\tau \|H(t)\| dt, \quad (39)$$

where $\|A\| = \sqrt{\text{Tr}[A^\dagger A]}$. Then, for any superadiabatic Hamiltonian $H_{SA}(t)$, we obtain

$$\Sigma(\tau) = \frac{1}{\tau} \int_0^\tau \sqrt{\sum_m [E_m^2(t) + \hbar^2 \mu_m(t)]} dt, \quad (40)$$

where $\{E_m(t)\}$ is the set of energies of the adiabatic Hamiltonian $H_0(t)$ and

$$\mu_m(t) = \langle \partial_t m(t) | \partial_t m(t) \rangle - |\langle m(t) | \partial_t m(t) \rangle|^2. \quad (41)$$

Equation (40) shows an increase in the energetic cost to superadiabatic evolutions compared to their adiabatic counterparts. Let us now evaluate the energetic cost to implement universal QC via teleportation. To this end, we calculate first the cost of single and double state teleportation and then extend the analysis for the cost of the implementation of quantum gates. By parametrizing the evolution in terms of the normalized time $s = t/\tau$, the energetic cost Σ_{single} for the teleportation of a single qubit reads as

$$\Sigma_{\text{single}} = \int_0^1 \sqrt{\sum_m \left[E_m^2(s) + \hbar^2 \frac{\mu_m(s)}{\tau^2} \right]} ds, \quad (42)$$

where $\mu_m(s) = \langle \partial_s E_m(s) | \partial_s E_m(s) \rangle$, which is a consequence of the fact that the set of eigenvalues of $H_0(s)$ is real. To illustrate the dependence of the energetic cost on the evolution path adopted, we will choose three interpolations: (i) linear interpolation, with $\eta_i(s) = 1 - s$ and $\eta_f(s) = s$; (ii) trigonometric interpolation, with $\eta_i(s) = \cos(\pi s/2)$ and $\eta_f(s) = \sin(\pi s/2)$; and (iii) exponential interpolation, with $\eta_i(s) = (e^{1-s} - 1)/(e - 1)$ and $\eta_f(s) = (e^s - 1)/(e - 1)$.

Then, we numerically evaluate the energetic cost as a function of $\omega\tau$ of by applying Eq. (42) to each interpolation, which is plotted in Fig. 4. In this plot, we explicitly show that the superadiabatic evolution recovers the cost of its adiabatic counterpart at the limit of infinite $\omega\tau$. Notice also that the usual linear interpolation is *not* the less costly option of interpolation. Moreover, the plot is in agreement with the energy-time complementarity relationship, with the faster evolutions costing more energy than slower dynamics. The energetic cost to implement the superadiabatic teleportation of an unknown n -qubit state can be provided in terms of the cost to implement the single teleportation as (see Appendix C)

$$\Sigma_n = g_n \Sigma_{\text{single}}, \quad (43)$$

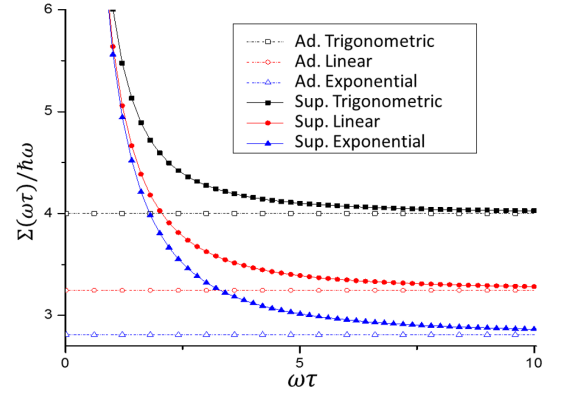


FIG. 4. (Color online) Energetic cost as a function of $\tau\omega$ for both adiabatic and superadiabatic dynamics of single qubit teleportation. Notice that the superadiabatic cost recovers the cost of its adiabatic counterpart in the limit $\tau\omega \rightarrow \infty$.

where we define the function $g_n = \sqrt{2^{3(n-1)}}n$. Moreover, the cost to implement *gate* teleportation of n qubits via superadiabatic evolution is also given by Eq. (43) due to the invariance of the Hilbert-Schmidt norm by unitary rotations. Note that the factor g_n exponentially increases with n . In any case, this is not a problem to perform universal QC with one and two qubits. In that case, we have $g_2 = 4$ and $g_3 = 8\sqrt{3}$, respectively.

IV. CONCLUSION

We introduced a general shortcut to the adiabatic gate teleportation model of quantum computation. Moreover, the model has been generalized to include the teleportation of an arbitrary n -qubit unitary gate. In particular, we have shown through Theorem 2 that the superadiabatic Hamiltonian for the teleportation of an n -qubit state can be directly used to implement the teleportation of an n -qubit gate U through a simple U rotation over the original superadiabatic Hamiltonian. As a main result of the work, we have shown that it is possible to devise fast local Hamiltonians to perform teleportation of one and two qubits as a primitive of universal QC. To analyze the energetic cost of the superadiabatic evolution, we considered the time-energy complementary relationship. In this context, it has been shown that the superadiabatic implementation is always more costly than its adiabatic counterpart, reducing to it in the limit of a long evolution time.

Implications of the superadiabatic approach applied to gate teleportation in a decohering environment is a further challenge of interest. In open systems, there is a competition between the adiabatic time scales, which require a long evolution, and the decoherence characteristic times, which require fast evolution. In this scenario, the superadiabatic implementation may provide a direction to obtain an optimal running time for the quantum algorithm while keeping an inherent protection against decoherence. A basis for such analysis may be provided by the generalization of the superadiabatic the-

ory for the context of open systems (see, e.g., Refs. [32–35]). The robustness of superadiabatic gate teleportation as well as experimental proposals are left for future research.

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Appendix A: Proof of the Theorem 1

The proof of Theorem 1 can be obtained as follows. If a time-dependent Hamiltonian $H_0(t)$ satisfies the commutation relation $[H_0(t), \Pi_z] = 0$, then we can write $[H_{SA}(t), \Pi_z] = [H_{CD}(t), \Pi_z]$. As Π_z and $H_0(t)$ have a common basis of eigenstates, an eigenstate $|n(t)\rangle$ of $H_0(t)$ has a definite Π_z parity so that we can write $\Pi_z |n(t)\rangle = (-1)^n |n(t)\rangle$ (by encoding the parity into the label n). By using $\Pi_z |\partial_t n(t)\rangle = (-1)^n |\partial_t n(t)\rangle$ it follows that $H_{CD}(t) \Pi_z = \Pi_z H_{CD}(t)$, thus implying $[H_{SA}(t), \Pi_z] = 0$. To complete the demonstration, from the hypothesis that $[H_0(t), \Pi_x] = 0$ is satisfied, we write $[H_{SA}(t), \Pi_x] = [H_{CD}(t), \Pi_x]$. Then, let us denote a matrix element of $[H_{CD}(t), \Pi_x]$ in the basis of eigenstates of $H_0(t)$ as

$$[H_{CD}(t), \Pi_x]_{kl} = \langle k(t) | [H_{CD}(t), \Pi_x] | l(t) \rangle. \quad (A1)$$

We now use that $\Pi_x |n(t)\rangle = |n'(t)\rangle$, where $|n(t)\rangle$ and $|n'(t)\rangle$ are eigenstates of the parity operator Π_z , with opposite eigenvalues. Moreover, $\Pi_x |\partial_t n(t)\rangle = |\partial_t n'(t)\rangle$. Then

$$(H_{CD}(t) \Pi_x)_{kl} = i\hbar [\langle k(t) | \partial_t l'(t) \rangle + \langle \partial_t l'(t) | l(t) \rangle \langle k(t) | l'(t) \rangle] = (\Pi_x H_{CD}(t))_{kl}. \quad (A2)$$

Thus $[H_{CD}(t), \Pi_x]_{kl} = 0 \forall (k, l)$. This proves Theorem 1.

Appendix B: Proof of the Theorem 2

In order to prove Theorem 2, consider two Hamiltonians $H(t)$ and $H(t, G)$ such that $H(t, G) = GH(t)G^\dagger$, with $GG^\dagger = \mathbb{1}$. The set of eigenvectors $|n(s), G\rangle$ of the Hamiltonian $H(t, G)$ can be determined from the set of eigenvectors $|n(s)\rangle$ of adiabatic Hamiltonian $H(s)$ as follows

$$|n(t), G\rangle = G |n(t)\rangle. \quad (B1)$$

Thus, the counter-diabatic Hamiltonian associated with $H(s, G)$ is given by

$$H_{CD}(s, G) = \frac{i\hbar}{\tau} \sum_n |\partial_s n, G\rangle \langle n, G| + \langle \partial_s n, G | n, G \rangle |n, G\rangle \langle n, G|. \quad (B2)$$

Then, by using Eq. (B1), we can show that

$$H_{CD}(s, G) = G \left[\frac{i\hbar}{\tau} \sum_n |\partial_s n\rangle \langle n| + \langle \partial_s n | n \rangle |n\rangle \langle n| \right] G^\dagger, \quad (B3)$$

where we have used that $|\partial_s n, G\rangle = G |\partial_s n\rangle$ and $GG^\dagger = \mathbb{1}$. Hence, we can write

$$H_{CD}(s, G) = GH_{CD}(s)G^\dagger. \quad (B4)$$

Eq. (B4) implies that $H_{SA}(t, G) = GH_{SA}(t)G^\dagger$. This proves Theorem 2.

Appendix C: Proof of Eq. (43)

In order to demonstrate Eq. (43), let us write the adiabatic Hamiltonian that is used to perform the n -qubit state teleportation as

$$H_{SA}(s) = \sum_{k=1}^n \mathcal{H}_k^{SA}(s) \quad (C1)$$

where $\mathcal{H}_k^{SA}(s) = (\otimes_{l=1}^{k-1} \mathbb{1}_l) \otimes H_k^{SA}(s) \otimes (\otimes_{l=k+1}^n \mathbb{1}_l)$, with $\mathcal{H}_k^{SA}(s)$ being a three-qubit Hamiltonian for each independent sector, as displayed in Fig. 3. Then, the energetic cost for the n -qubit superadiabatic teleportation reads as

$$\Sigma_n = \int_0^1 ds \sqrt{\text{Tr}[H_{SA}^2(s)]}, \quad (C2)$$

where we can write

$$H_{SA}^2(s) = \sum_{k=1}^n [\mathcal{H}_k^{SA}(s)]^2 + \sum_{m \neq k} \left(\sum_k \mathcal{H}_k^{SA}(s) \mathcal{H}_m^{SA}(s) \right). \quad (C3)$$

Now, we use that, for $k \neq m$, we get

$$\text{Tr}[\mathcal{H}_k^{SA}(s) \mathcal{H}_m^{SA}(s)] = (\text{Tr}[\mathbb{1}])^{n-2} \text{Tr}[H_k^{SA}(s)] \text{Tr}[H_m^{SA}(s)]. \quad (C4)$$

Then, we write $\text{Tr}[H_j^{SA}(s)] = \text{Tr}[H_j^{(0)}(s) + H_j^{CD}(s)]$, where $H_j^{(0)}(s)$ is the original (adiabatic) Hamiltonian at sector j and $H_j^{CD}(s)$ its corresponding counter-diabatic Hamiltonian. By explicitly computing the trace in the eigenstate basis of $H_j^{(0)}(s)$ and by using Eqs. (9)-(11) and (29), we obtain that $\text{Tr}[H_j^{SA}(s)] = 0 \ (\forall j \in \{1, \dots, n\})$, which implies

$$\text{Tr}[\mathcal{H}_k^{SA}(s) \mathcal{H}_m^{SA}(s)] = 0 \quad (k \neq m). \quad (C5)$$

Thus, the energetic cost for the n -qubit state teleportation reads as

$$\begin{aligned} \text{Tr}[H_{SA}^2(s)] &= \sum_{k=1}^n \text{Tr}[\{\mathcal{H}_k^{SA}(s)\}^2] \\ &= (\text{Tr}[\mathbb{1}])^{n-1} \sum_{k=1}^n \text{Tr}[\{H_k^{SA}(s)\}^2] \\ &= 2^{3(n-1)} n \text{Tr}[\{H_k^{SA}(s)\}^2] \quad (\forall k). \end{aligned} \quad (C6)$$

Hence, Eq. (C6) into Eq. (C2) yields

$$\Sigma_n = \sqrt{2^{3(n-1)}} n \Sigma_{\text{single}}, \quad (\text{C7})$$

which proves the validity of Eq. (43).

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